Beauville + Voisin. On the Chow ring of a K3 surface.

Theorem. X K3

$$\Rightarrow \quad c_2(X) = 24\, c_X \in CH_0(X)$$

Lemma: $x, y \in E$ elliptic curve

$$\Rightarrow (x,x) - (x,y) - (y,x) + (y,y) = 0 \in CH_0(E \times E)$$

(formally, $\xi := (x-y, x-y) = 0$)

Pf: $\mathrm{Sym}^2 E \xrightarrow{a=+} E$ is $\mathbb{P}^1$-bundle.

$$\Rightarrow CH_0(\mathrm{Sym}^2 E) \underset{\cong}{\xrightarrow{a_*}} CH_0(E)$$

$$\eta := (x,x) + (y,y) - 2(x,y) \mapsto [2x] + [2y] - 2[x+y] = 0$$

$$\Rightarrow \quad \eta = 0 \in CH_0(\mathrm{Sym}^2 E)$$

$$\begin{array}{ccc} \eta & & CH_0(\mathrm{Sym}^2 E) \\ \downarrow & & \downarrow \text{pullback} \\ 2\xi & & CH_0(E \times E) \end{array}$$

• $\left.\begin{array}{l} 2\xi = 0, \quad \deg \xi = 0 \\ [\xi] = 0 \in \mathrm{Alb}(E \times E) \end{array}\right\} \xRightarrow{\text{Rojtman}} \xi = 0$

Prop: $X \xhookrightarrow{\Delta} X \times X \rightsquigarrow CH_i(X) \xrightarrow{\Delta_*} CH_i(X \times X)$

$$\begin{array}{ccc} & X \times X & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi} \\ X & & X \end{array}$$

a) On CH_1, $\Delta_* \alpha = \alpha \times c_X + c_X \times \alpha$

b) On CH_0, $\Delta_* \xi = \xi \times c_X + c_X \times \xi - (\deg \xi)\, \Delta_* c_X$

$\mathfrak{X} \in CH_2(X\times X\times X)_{\mathbb{Q}}$ (w/ fixed $c \in \mathbb{P}^1 \subset X$)

$$:= \Delta_{123} - c\,\Delta_{23} - \overset{c}{\underset{\Delta_{13}}{\times}} - \Delta_{12}\, c + X c c + c X c + c c X$$

$$= (x,x,x) - (c,x,x)_{+c.c.} + (x,c,c)_{+c.c.}$$

$$\left(\approx \bigcup_x (x-c)^3 + \underbrace{c^3}_{\in CH_0}\right)$$

Prop. $\mathfrak{X} = 0$

Cor. $\forall\, \xi \in CH_2(X\times X) \xrightarrow{\Delta^*} CH_0(X)$

$$\Rightarrow \xi\cap\Delta - \xi\cap(X\times c) - \xi\cap(c\times X) \in \mathbb{Z}c$$

In particular, take $\xi = \Delta$

$$\Delta\cap\Delta = c_2(X) \qquad (\because N_{\Delta/X\times X} = T_X)$$

$$\Delta\cap(X\times c) = c = \Delta\cap(c\times X)$$

$$\Rightarrow \quad c_2(X) \in \mathbb{Z}c \subset CH_0(X).$$

Proof [Prop $\Rightarrow$ Cor]

$$(X\times X)\times X \supset \mathfrak{X}, \qquad \xi \in X\times X \xleftarrow{p_{12}} (X\times X)\times X \xrightarrow{p_3} X$$

$$p_{3*}(p_{12}^*(\xi)\cdot \Delta_{123}) = \Delta^*\xi \in CH_0(X)$$

$$\left((p,q,X)\cap \textstyle\bigcup (x\times x) \rightsquigarrow p=q \xrightarrow{p_3} x=p=q\right)$$

$$p_{3*}(p_{12}^*(\xi)\cdot c\,\Delta_{23}) = \xi\cap(X\times c)$$

$$p_{3*}(p_{12}^*(\xi)\cdot \Delta_{12}\, c) \in \mathbb{Z}c$$

$$p_{3*}(p_{12}^*(\xi)\cdot \underbrace{c\,c\,X}_{p_{12}^*(c\cdot c)}) = 0$$

§ Proof of Prop

Approach: $CH_2(X^3)_{\mathbb{Q}} \longrightarrow H_4(X^3, \mathbb{Q})$

$\mathfrak{X} \longmapsto 0$ (direct inspection).

"If" $\mathfrak{X} = C \cdot [\text{effective cycle}]$, then $\mathfrak{X} = 0$.

Will reduce to the following properties for elliptic curves:

F elliptic curve

$\mathrm{Pic}(F^3)_{\mathbb{Q}} \supset \mathrm{Pic}(F^3)^{inv} \leftarrow$ inv. under permutations ($S_3 \times \mathbb{Z}_2$) and involution (-1_{F^3}).

Lemma 2.6

a) $\nu := (u,u,u) - (e,u,u)_{+c.c.} + (u,e,e)_{+c.c.} = 0 \in CH_1(F^3)_{\mathbb{Q}}$

$= \bigcup_{u \in F} (u-e)^3 + e^3$ (e ← origin of F)

b). $\mathrm{Pic}(F^3)^{inv} = \mathbb{Z}\langle \alpha_F \rangle + \mathbb{Z}\langle \beta_F \rangle$ where

$\alpha_F := e \times F \times F + F \times e \times F + F \times F \times e$

$\beta_F := \Delta_{12} \times F + \underset{\Delta_{13}}{\overset{F}{\times}} + F \times \Delta_{23}$

i.e. $\alpha_F = (e \times F \times F)_{+c.c.}$ and $\beta_F = (\Delta_{12} \times F)_{+c.c.}$

Pf. of a) ν in $S^3F \xrightarrow[+]{\mathbb{P}^2} F$, let $h \overset{div.}{\subset} S^3F$ $[h]|_{\mathbb{P}^2} = [\mathbb{P}^1]$.

$\cup$

Image $(e \times F \times F) =: h$, $[h]|_{\mathbb{P}^2} = [\mathbb{P}^1]$ $(\because S^2F \xrightarrow{\mathbb{P}^1} F)$

$\nu \in CH_1(S^3F) = \mathrm{Pic}(F) \cdot h + \mathbb{Z}\langle h^2 \rangle$

write $\nu = d \cdot h + \cancel{n}\, h^2$

$(\because n = \deg(\nu \cdot \mathbb{P}^2) = 3^2 - 3 \times 2^2 + 3 \times 1 = 0)$

$\nu \cdot (e \times F \times F)$ in $F \times F \times F$

$= \big((uuu) - (euu) - (ueu) - (uue) + (uee) + (eue) + (eeu)\big) \cdot (eFF)$

$= (eee) - \cancel{(euu)}^{1D} - (eee) - (eee) + (eee) + \cancel{(eue)}^{1D} + \cancel{(eeu)}^{1D} = 0$

$\Rightarrow d = 0 \Rightarrow \nu = 0$.

Pf. of b). $\mathrm{Pic}(S^3F) = \mathrm{Pic}(F) + \mathbb{Z}\langle h \rangle$

$\Rightarrow \mathrm{Pic}(F^3)^{inv} = \mathrm{Pic}(S^3F)^{(-1_{F^3})}$ has rank 2.

(e.g. $\mathrm{Pic}(F) \circlearrowleft (-1_F)$ acts trivially).

Need α_F, β_F indep. in $\mathrm{Pic}(F^3)$. ✓

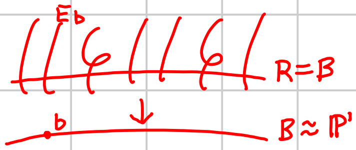
$|_{F^2 \times 0} \rightsquigarrow F \times e + e \times F\ ,\ \Delta + F \times e + e \times F$ indep. $\Rightarrow$

For simplicity, $\mathrm{Pic}(X) = \mathbb{Z}\langle H\rangle$ (otherwise, by taking limits.)

$|H| \ni E_b'$ (possibly singular) elliptic curve, $b \in B$ ← 1 dim.

$\rightsquigarrow$ $E \xrightarrow[\text{generic finite}]{\pi} X$ s.t. $p^{-1}(b) =: E_b \xrightarrow[\text{birat.}]{\pi} E_b'$, with elliptic fibration $p: E \to B$

- $\exists$ section $s : B \to E$ (Taking covering of B)
- $\pi \circ s(B) \subset X$ rational curve. $\left(\begin{array}{l}\text{Take } X \supset R \text{ rat}^l \text{ curve}, \not\supseteq E_b' \ \forall b \\ B = \pi^{-1}(R)_{\text{irred. comp.}}\end{array}\right)$
- $\forall E_b$ irred. ($\because \mathrm{rk}(\mathrm{Pic}) = 1$)

Simplest case $X = E \longrightarrow B$  $X = E.$

- $E_B^3 := E \times_B E \times_B E \xrightarrow{\pi^3} X^3$

$\mu \in CH_2(E_B^3)$

$:= (u,u,u) - (sp(u), u, u)_{+c.c.} + (u, sp(u), sp(u))_{+c.c.}$

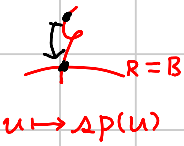
$\left(\approx \bigcup_{u \in E/B} (u - s)^3 + s^3 \right)$

Claim: $\pi^3_*(\mu) = (\deg \pi) \cdot \mathfrak{X}$.

Say $E \xrightarrow[\simeq]{\pi} X$

1) $\pi^3_*(u,u,u) = (x,x,x) \in CH_0(X^3)$ ($\because sp(u) \in \mathbb{P}^1$)

2) $\pi^3_*(u, u, sp(u)) = (x,x,c) + a\,(\Delta_* E_b') \times E_b'$

3) $\pi^3_*(u, sp(u), sp(u)) = (x, c, c) + a\, E_b' \times (\Delta_* E_b')$ 

For 2) $\pi^3_*(u, u, sp(u)) = p_{12}^* \Delta \cdot p_{23}^* \Gamma$ w/ $\Gamma \triangleq \mathrm{Graph}(X \to X)$

$\Gamma \overset{\text{divisor}}{\subset} X \times R \subset X^2 \Rightarrow \Gamma \equiv \underbrace{D \times R}_{a E_b' \times E_b'} + X \times c$ $\exists D \overset{\text{div.}}{\subset} X$ ($\because \mathrm{Pic}(X) = \mathbb{Z}\langle E_b'\rangle$)

After permutat^n, those involving a's cancel.

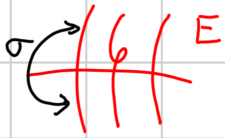
Claim: $\pi^3_*(\mu) = C\,\mathfrak{z} \quad \in CH_2(X^3)$.

where $\mathfrak{z} = (c \times E_b' \times E_b')_{+c.c.}$ (effective)

Pf. of claim: Restrict to generic fiber of $E^3_B \to B$:

$$\mu|_{E_b^3} = \nu \overset{\text{Lemma a)}}{=\!=\!=} 0$$

$$\Rightarrow \quad \mu = \sum_{b_i} D_{b_i} \quad \text{w/ } D_{b_i} \overset{\text{div.}}{\subset} E^3_{b_i}$$



, $\sigma|_{\text{fiber}} = -1_{E_b} \rightsquigarrow \sigma^3 : E^3_B \circlearrowleft$ involut^n

$$\rightsquigarrow \quad S_3 \times \mathbb{Z}_2 \curvearrowright E^3_B .$$

$$\mu : S_3 \times \mathbb{Z}_2\text{-inv.} \Rightarrow D_{b_i} \in CH_2(E^3_{b_i})^{inv}_{\mathbb{Q}}$$

$$\overset{\text{Lemma b)}}{\Longrightarrow} \quad D_{b_i} \in \mathbb{Z}\langle \alpha_{E_{b_i}}, \beta_{E_{b_i}} \rangle$$

- $\alpha_{E_{b_i}} = (e_{E_{b_i}} \times E_{b_i} \times E_{b_i})_{+c.c.} \overset{\pi^3_*}{\longmapsto} \mathfrak{z} \in CH_2(X^3)$

  (∵ $\pi(e) \in R \subset X$ rational curve $\Rightarrow e_{E_{b_i}} = c_X$)

- $\beta_{E_{b_i}} = (\Delta_{E_{b_i},12} \times E_{b_i})_{+c.c.} \overset{\pi^3_*}{\longmapsto} 2\,\mathfrak{z}$

  (∵ Prop a): $\Delta_*\alpha = \alpha \cdot c_X + c_X \cdot \alpha \in CH_1(X^2)$

  reason: WLOG $\alpha \simeq \mathbb{P}^1$, $\Delta_{\mathbb{P}^1} = \mathbb{P}^1 \times 0 + 0 \times \mathbb{P}^1 \in CH_1((\mathbb{P}^1)^2)$)